Introduction

The generalized gamma (GG) formulation for representing hydrometeor particle size distributions (PSD) has been considered by several researchers in the past several decades, ranging from cloud droplet and fog PSDs (Deirmendjian 1969; Hess et al. 1998; Vivekanandan et al. 1999, also referred to as modified gamma distribution) to ice PSDs (Delanoe et al., 2005, Petty and Huang, 2011). For rain, the use of GG formulation has been somewhat more recent (Auf der Maur, 2001, Lee et al., 2004, Kuo et al., 2004, Raupach and Berne, 2017, Thurai and Bringi 2018).

From theory, the GG form is mathematically sound and has the following desired properties: (a) it reduces to standard gamma, exponential, and Weibull forms (which are special cases) and as a limiting case to the log-normal form, (b) any power law of drop diameter (D) such as mass or mass flux also falls in the GG form and (c) it maximizes relative entropy under moment constraints (Wu and McFarquhar, 2018). As shown in the latter, it is very difficult from measurements alone to prove via statistical methods that the GG is optimal compared to standard gamma (SG) or log-normal forms. While the SG distribution is widely adopted in radar rainfall studies as well as in multi-moment numerical schemes, some studies have shown via rigorous statistical (Kolmogorov-Smirnov) goodness-of-fit tests that around 50% or more of 1-min measured DSDs reject the SG model as a fit to the data (Adirosi et al. 2014).

Following Auf der Maur (2001), in the GG formulation, the rain drop size distribution (DSD), represented by N(D), can be expressed as:

\[ N(D) = M_0 \frac{c \lambda}{\Gamma(\mu_{GG})} (\lambda D)^{\mu_{GG}-1} \exp[-(\lambda D)^c] \]  

(1)

where \( M_0 \) is the zeroth moment (= total number of drops per m\(^3\)) and \( \mu_{GG} \) and \( c \) are two shape parameters, and \( \lambda \) can be expressed in terms of two reference moments, \( M_i \) and \( M_j \) (Lee et al., 2004), leading to the double-moment normalized version of eq. (1), given by:

\[ N(D) = N_0' h_{GG(i,j;\mu_{GG}, c)}(x) \]

(2)

where \( N_0' = M_0 (j+1)/(j-i) M_j (i+1)/(i-j) \) and \( D_m' = (M_j / M_i)^{1/(j-i)} \). \( h_{GG(i,j;\mu_{GG}, c)}(x) = c \Gamma_i^{\mu_{GG}+\frac{i}{c}} \Gamma_{(j-i)+\mu_{GG}}^{(i+1)/(i-j)} x^{\mu_{GG}-1} \exp[-\Gamma_i \Gamma_{j-i} x^c] \) and \( x = \left( \frac{D}{D_m'} \right)^{\frac{1}{c}} \)

If we set \( i=3 \) and \( j=4 \), then \( D_m' \) becomes equal to \( D_m \), the mass-weighted mean diameter. Note also that eq. (2) will reduce to the SG model (for \( c=1 \)) defined by the three parameters \( [N_0, D_0 \) or \( D_m \) and \( \mu_{SG} \)] where \( \mu_{SG} = \mu_{GG} - 1 \) and \( N_0 \), the normalized intercept parameter, is simply related to \( N_0' \) via \( N_0 = (4/6)N_0' \). Furthermore, eq. (2) further reduces to an exponential distribution when \( \mu_{SG} = 0 \) (or \( \mu_{GG} = 1 \)).

The DSD formulation used herein is also based on double-moment normalization using the 3\(^{rd} \) and 4\(^{th} \) moments as the reference moments (\( M_3,M_4 \)) and the intrinsic shape of the normalized distribution h(x) where x=D/D\(_{43}\) is the scaled diameter and D\(_{43}\) (i.e., D\(_m\)), i.e. the ratio of M\(_4\) to M\(_3\). N(D) can be expressed in compact form as N(D)=M\(_3\)/D\(_{43}\)\(^3\) h(x; \( \mu_{GG}, c \))=N\(_0\) h(x; \( \mu_{GG}, c \)). This form has the advantage that any moment of the DSD can be expressed as power laws of the two reference moments and (\( \mu_{GG}, c \)); in particular the lower order moments (\( M_0, M_1, M_2 \)) which are involved in modeling microphysical processes such as coalescence and breakup which control the DSD shape at the small drop and large drop ends as well as the plateau region in between. Most currently available disdrometers cannot accurately measure the small drop end (D<0.5-0.7 mm) due to poor resolution and sensitivity issues.
2. Data

We consider here measurements of rain drop size distributions (DSD) using collocated high resolution (50 microns) Meteorological Particle Spectrometer (MPS) and moderate resolution (170 microns) 2D-video disdrometer from two different rain climatologies (Greeley, Colorado, and Huntsville, Alabama, USA). In both locations, the instruments were installed within a DFIR double wind-fence in order to mitigate wind effects, especially for small drops. The combination of data from the two instruments yields what we term as the “full” or composite DSD spectra, e.g. over 1-minute, covering the size range from 100 microns to large rain drops whose shapes were found to be not very well represented by the standard gamma model, but required the additional flexibility offered by eq. (2), with the two abovementioned shape parameters ($\mu_{GG}$, c). We define the composite spectra as using the MPS for $0.1<D<1$ mm and 2DVD for $D > 1$ mm based on resolution and sample volume considerations. An example is given below in Fig. 1 where the 2DVD data are shown in blue, the MPS data in black and the composite DSD in green. Also shown is the fitted GG curve in red, where the estimation of $\mu$ and c is done with a global search using nonlinear least squares, as mentioned in Thurai and Bringi, (2018). Note also that we allow $\mu$ to go negative to capture the drizzle mode (Raupach et al., 2018a, 2018b).

Fig. 1: An example of a 3-minute DSD from the MPS and the 2DVD, together with the composite DSD and the fitted GG to eq. (2).

Examples of events presented here include light, shallow rain, stratiform rain, and convective rain events in Greeley, as well as outer bands of hurricane Irma and tropical storm Nate. The latter two are special cases, which fortuitously had crossed our instrumentation site in Huntsville (HSV).

2 Greeley examples

The first two examples from Greeley are taken from long duration but intermittent rain event, which was part of a mid-latitude synoptic-scale cyclone that produced different types of rain ranging from light stratiform to strong convective (Thurai et al., 2017). The set of plots in panel (a) in Fig. 2 show the 3-min DSD measurements (09:45 - 10:00 UTC) during light precipitation. Superimposed is their corresponding generalized gamma fitted curves in red. Values of $D_m$ for these cases were ~ 0.5 mm, which is expected for DSDs mostly consisting of small drops (maximum drop diameters < 1.2 mm). CSU-CHILL S-band radar RHI scan made over the instrument site (panel (b)) shows around 10 dBZ at 13 km range to the instrumented site. The rain system was shallow with low echo tops and no bright-band. LDR (linear depolarization ratio in panel (c)) does not show any bright-band either. The drizzle mode is clear for $D<0.5$ mm with a distinct concave upwards shape; this mode causes the $D_m$ to be smaller and the spectral width to be larger than DSDs measured only by 2DVD as described in Thurai et al. (2017).
Fig. 2: (a) Light rain DSDs during the 17 April 2015 event at Greeley (GXY). (b) dBZ from CHILL S-band RHI over the disdrometer site (13 km range); (c) the corresponding LDR. In (a) MPS data are in black, 2DVD data in blue and the GG fitted curve in red.

The second example, shown in Fig. 3 is taken during stratiform rain period at the disdrometer site (on the same day). Panel (a) shows two 3-minute DSD measurements at 13:21 to 13:27 UTC. Somewhat larger drops were recorded, and $D_m$ values were around 1 and 0.8 mm respectively. Once again, the drizzle mode is evident. The CHILL S-band RHIIs over the disdrometer site at 13:23 UTC are shown in panels (b) and (c). Very clear bright band can be seen from the LDR plot.
The third example is from a convective event which occurred on 10 Aug 2015 and which lasted for about 35 minutes over the disdrometer site. Panel (a) shows two 3-minute DSD measurements at 21:57 to 22:03 UTC. Significantly larger drops were recorded ($D_{\text{max}} > 4.5$ mm), and $D_m$ values were around 2.4 and 2.1 mm respectively. Once again, the drizzle mode is evident. The CHILL S-band $Z$ over the disdrometer site taken at 22:02 UTC is shown in panel (b) and the corresponding $Z_{dr}$ is shown in panel (c). 40-45 dBZ values can be seen at 13 km range, and ~2 dB $Z_{dr}$, being consistent with the large drops seen in panel (a).

In all three cases, i.e. Fig. 2, 3, and 4, the fitted GG curves are seen to closely represent the composite DSDs. A further example can be seen in Thurai et al., (2017b), which shows animation of a 90-minute event on 23 May 2015 during which the 3-minute rainfall rate reached a maximum of over 60 mm/h. There too the fitted curves can be seen to closely represent the composite DSD data throughout the 90 minute period.

**Fig. 3**: (a) Stratiform rain DSDs during the 17 April 2015 event at Greeley (GXY). (b) dBZ from CHILL S-band RHI over the disdrometer site (13 km range); (c) the corresponding LDR. In (a) MPS data are in black, 2DVD data in blue and the GG fitted curve in red.
3 Huntsville examples

3.1 Outer bands of Hurricane Irma

On 11-12 September 2017, the outer rain bands of Hurricane Irma produced rainfall for more than 8 h over our instrumented site at UAH leading to unique data sets of the evolution of the drop size, shape, and fall speed distributions over the entire size range. The radar mosaic map of the Irma outer bands at 00:55 UTC on 12 Sept 2017 is shown in Panel (a) of Fig. 5. Collocated with the MPS and 2DVD (inside the DFIR as mentioned earlier) was a Precipitation Occurrence Sensor System (POSS, Sheppard and Joe, 2008). A 1-hour DSD comparison between the composite DSD and the POSS-based (calibrated) DSD is shown in panel (b). The agreement is very close, and the rise in the drop concentration in the sub-mm region is also evident in both datasets.

Fig. 5(a): Radar mosaic image over SE US at 00:55 UTC on 12 Sept. 2017 (University of Alabama, Huntsville, UAH, is marked with an arrow); (b) a 1-hour DSD comparison between the MPS+2DVD (SN16) composite DSD and (calibrated, collocated) POSS-based DSD during the passage of Hurricane Irma outer bands, from 00 – 01 hr UTC on 12 Sept 2017.

Fig. 4: (a) Convective rain DSDs during the 10 Aug 2015 event at Greeley (GXY). (b) dBZ from CHILL S-band RHI over the disdrometer site (13 km range); (c) the corresponding Z_{dr}. In (a) MPS data are in black, 2DVD data in blue and the GG fitted curve in red.
Four examples of 3-minute averaged “composite” DSDs constructed from MPS and 2DVD measurements on 12 Sept 2017 are shown in Fig. 6(a-d) in green, along with their fitted GG model in red. Once again, the fitted curves represent the measured data very well.

Panel (e) shows the reflectivity-height profiles from the UAH-XPR (after calibration) vertically-pointing X-band Doppler radar, from 00:00 to 06:00 UTC. Melting layer is evident throughout the event but with variable thickness and intensity. Panels (f) and (g) show the $N_w$ and $D_m$ variations derived from the 1-minute composite DSDs for 00 – 06 hr UTC. Some variation in $D_m$ can be seen; during the thick bright-band period of 03-04 hr, $D_m$ values reach 1.5 mm whereas during weak rain period after 05:30 UTC, they go below 1 mm. Fitted values of $[\mu_{GG}, c]$ are shown in panels (h) and (i), respectively. The shape parameter $\mu_{GG}$ appears to be relatively steady whereas the second shape parameter $c$ exhibits somewhat more variability, attaining some of its lowest values during the thick BB period. Panel (d), at 03:15 UTC shows that $N(D)$ of $10^{-2}$ /mm/m$^3$ from the fitted GG corresponds to the highest drop diameter (~4.5 mm) amongst the four panels.

Fig. 6: (a) to (d): Four examples of 3-minute composite DSDs (light green) from Irma outer bands, along with their GG fits; (e) Vertical profiles of reflectivity from UAH-XPR Doppler radar (courtesy of Prof. Knupp et al., at UAH); (f) $N_w$ from the ground-based 1-minute composite DSD; (g) the corresponding $D_m$; (h) the fitted $\mu$, and (i) the fitted $c$ values.
3.2 Tropical storm Nate

Fig. 7 shows an example of shallow or weak rain, which was part of the tropical storm Nate that had also traversed our instrument site at UAH. Panels (a) and (b) show, respectively, the Z and mean Doppler velocity versus time from the XPR for 17-18 UTC. A weak bright-band is noted near 5 km with echo tops < 5.6 km height, which is apparent in the mean Doppler velocity starting at 1700 UTC when the DSD shows a drizzle mode (panel c). After 1754 UTC, the echo tops are very shallow and entirely warm rain; the DSD now has a distinctly different shape at the small drop end (convex down). While these echoes are from the remnants of Nate after landfall, nevertheless they could have some oceanic characteristics. Martner et al. (2008) report on such shallow echoes without bright-band as well as echoes with BB and note very distinct DSDs between the two (In general however, oceanic rain tend to have more dominant component of small drops concentration, for example, Thompson et al. 2015).

Fig. 7: (a) XPR observations of dBZ (time-height cross-sections) during tropical storm Nate; (b) mean Doppler velocity (courtesy of Dustin Conrad and Prof. Kevin Knupp); (c) DSDs during weak bright-band, low Z/R, period (in pink box) and shallow warm rain (in blue box). The composite DSDs are shown as green curve in panel (c). As in Fig. 1(a), the GG curves show excellent fits to the composite DSDs.

Another time period during Nate is shown in Fig. 8. The middle panel shows the XPR Z plot for 1500-1600 UTC. Some examples of the 3-min composite DSDs are shown surrounding panels whose times are ‘connected’ to the center panel with blue lines. At 15:24 there is some discrepancy between the MPS and 2DVD data, which corresponds to the highly bent vertical profiles of Z. This observation is noteworthy. It is possible that unusual size sorting may be occurring during this time period. It also appears that there may be some growth occurring in the lowest 1-km. Other cases show (as before) very close representativeness of the GG fitted curves to the data.

Fig. 8: 15:00-16:00 UTC during Nate. Center panel shows the dBZ height profiles from UAH-XPR and the surround panels show examples of 3-minute DSDs at various times.
4 Overall assessment

In section 1 we referred to the compact representation for \( N(D) = N_0 h(x) \), as shown by Lee et al. (2004; their eq. (43), and based on earlier work by Testud et al., 2001). In our composite DSD datasets, we have seen that \( h(x) \) can be best represented in terms of the generalized gamma form with the two shape parameters (\( \mu_{GG} \), \( c \)) so that it can simultaneously fit the small drop and large drop ends together with a ‘shoulder’ region in between (Thurai and Bringi, 2018). The stability of \( h(x) \) has been demonstrated using Parsivel disdrometer data for stratiform rain by Raupach and Berne (2017), but with the caveat that the small drop end could not be accurately characterized. Here we show the stability of \( h(x) \) at the Greeley site (GXY) in Fig. 9, panel (a) using contoured frequency of occurrence plots of \( h(x) = N(D)/N_W \) versus \( x = D/D_m \) for a large number of 1-minute DSDs during rain-only events. Overlaid onto the plot is the optimized, or most probable, GG fit where the small drop end (actually \( x<0.5 \)) has a concave up shape while the large drop end (\( x>2 \)) falls faster than an equivalent exponential (e.g., due to drop break-up). Histograms of the fitted values of the two shape parameters are shown in panels (b) and (c). Whereas \( \mu \) shows a narrow histogram, mostly within -1 to +0.2, values of \( c \) show wider distribution. In panel (b), the standard gamma value for \( c=1 \) is marked in magenta. Clearly, it is not the most probable value. In fact, in Thurai and Bringi (2018), we show in Table 2, that less than 17% of the cases had \( c \) values in the range 0.5 to 1.5. Moreover, heuristic goodness-of-fit methods were used to demonstrate that the generalized gamma model outperforms the SG model which has only one shape parameter.

![GXY: Rain only cases](image)

Fig. 9: (a) \( h(x) \) versus \( x \) in terms of frequency of occurrence (color) and the most probable variation (black curve); (b) and (c) histograms of fitted \( c \) and \( \mu \) values from a large number of 1-minute DSDs from rain-only events at Greeley.

To illustrate the goodness of fit, in Fig. 10 we show histograms of the normalized bias for various drop diameter intervals. For each histogram, the center diameter (in mm) is shown on top. For the small drops, diameter interval in 0.05 mm (MPS resolution), and for the larger drops (> 0.75 mm), the diameter interval was set to 0.25 mm. In most cases, the bias appear to be fairly close to 0, and histograms are Gaussian-like, but larger diameters have broader histograms because of less no. of points, which is to be expected. The total number of 1-minute points, altogether, was 12177, taken from both GXY and HSV events.

Finally in Fig. 11 we show the variation of \( D_m \) versus \( \sigma_M \), the standard deviation of mass spectrum, derived for the same 12177 1-minute DSDs. The color intensity plot represents the frequency of occurrence corresponding to the data itself and those derived from the GG fits are shown as black dots. Most of these points lie well within the ‘higher frequency region’, giving further confirmation of the high representativeness of the GG fits. We have also included three curves, one in black and the other two in gray. They represent the equations (22) – (24), given in Williams et al. (2014), with the black curve showing the most likely variation and the gray lines the upper and lower bounds. The underestimation of \( \sigma_M \) from these curves is largely due to lack of the small drops (MPS) data being included in their DSD data.

5. Conclusions

We believe that we have answered the question in the title ‘Why Generalized Gamma” not necessarily via statistical testing but based on fidelity of the visual fits to our composite DSD data, especially for those which have double curvature. Also when \( \mu_{GG} \) is slightly negative, it controls the shape at the small drop end simultaneously with \( c \) controlling the shape at the large drop end. This, the standard gamma with \( c=1 \) and negative \( \mu_{SG} \) cannot do. Although the generalized gamma has been shown to better represent the full DSD spectrum, additional research is required on the sensitivity of its shape parameters, especially \( c \), to rainfall type and meteorological regime in order to apply this DSD model to existing radar-based rainfall retrieval algorithms.
Fig. 10: Histograms of the normalized bias for various drop diameter range. The center diameter is on top of each panel.

Fig. 11: $D_m$ versus $\sigma_M$ from 7435 1-minute DSDs (color intensity) compared with the corresponding GG-fitted variations. The black line is eq. (22) of Williams et al (2014) and the grey lines are their eq. (23) and eq. (24).

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